Combinatorial Networks
Week 2, March 18-19

Partially ordered set

Let X be a finite set.

- Definition. R is called a *relation* on set X, if $R \subset X \times X$.

  Here $X \times X := \{\text{all ordered pairs } (x,y) : x, y \in X\}$ denotes the Cartesian product.

- If $(x,y) \in R$, then we will write it as xRy.

- **Definition.** A *partially ordered set* (or *poset* for short) is an odered pair (X,R), where X is a finite set and R is a relation on X that satisfies the following properties:

  (1) R is *reflexive*: xRx for any $x \in X$;

  (2) R is *antisymmetric*: xRy and yRx imply that $x = y$. (In other words, for distince $x, y \in X$, at most one of xRy, yRx can occur.)

  (3) R is *transitive*: xRy and yRz imply that xRz.

- Example. Consider the poset (X,R), where $X := 2^{[n]}$ and relation R is defined according to the inclusion relationship: if $A \subset B$, then $(A,B) \in R$ or ARB.

  We will express this important poset as $(2^{[n]}, \subset)$ from now on.

- We often use "$\preceq$" to replace relation "R", when this ordering $\preceq$ is clear from the context. For example, $x \preceq y$ means xRy; poset $(X, \preceq)$ is just (X,R).

  We write $x \prec y$, if $x \preceq y$ and $x \neq y$.

- If $x \prec y$, then x is called an *predecessor* of y.

- We say $x \in X$ is a *minimal element* of poset $(X, \preceq)$, if there is no predecessor of x.

- **Definition.** Let $(X, \preceq)$ be a poset. We say element x is an *immediate predecessor* of element y, if

  (i) $x \prec y$, and

  (ii) there is no element $t \in X$ such that $x \prec t \prec y$.

  If x is an immediate predecessor of y, then we write it as $x \lhd y$. Note that $x \lhd y$ does imply that $x \prec y$.

- **Fact.** For any two distinct elements x, y in poset $(X, \preceq)$, $x \prec y$ if and only if there exist finite many elements $x_1, x_2, ..., x_k \in X$ such that $x \lhd x_1 \lhd x_2 \lhd \ldots \lhd x_k \lhd y$.

  Note that k can be equal to 0 here.

- **Proof.** One direction is easy: if $x \lhd x_1 \lhd x_2 \lhd \ldots \lhd x_k \lhd y$, then $x \prec x_1 \prec x_2 \prec \ldots \prec x_k \prec y$, by transitivity we get $x \prec y$.

  For any pair $x \prec y$, define $M_{xy} := \{t \in X : x \prec t \prec y\}$ to be a subset of X.

We prove "the other direction" by induction on the size of M_{xy}. Consider a pair $x \prec y$. Base case: if $|M_{xy}| = 0$, then this means there is no $t \in X$ such that $x \prec t \prec y$, therefore $x \lhd y$ and this corresponds to the case $k = 0$.

Now assume that "the other direction" holds for all pairs $a \prec b$ with $|M_{ab}| < n$. Let $x \prec y$ with $|M_{xy}| = n \geq 1$. Pick any $t \in M_{xy}$, then $x \prec t \prec y$.

We consider M_{xt} and M_{ty}. It is not hard to see that $M_{xt} \subset M_{xy} - \{t\}$ and $M_{ty} \subset M_{xy} - \{t\}$ by transitivity. So $|M_{xt}| < n, |M_{ty}| < n$, then we are able to apply induction on pairs $x \prec t$ and $t \prec y$. By induction, there exists finite many elements $x_1, ..., x_k, y_1, ..., y_l$ such that $x \lhd x_1 \lhd \ldots \lhd x_k \lhd t$ and $t \lhd y_1 \lhd \ldots \lhd y_l \lhd y$, implying that

$$x \lhd x_1 \lhd \ldots \lhd x_k \lhd t \lhd y_1 \lhd \ldots \lhd y_l \lhd y.$$

This completes the proof. ∎

- One of the nice properties about poset is that we can express any poset in a digram!

  Definition. The *Hasse digram* of a poset $(X, \preceq)$ is a drawing in the plane such that

  (i) each element of X is drawn as a node, and

  (ii) each pair x, y with $x \lhd y$ is connected by a line segement, and

  (iii) if $x \lhd y$, then the node x appears lower in the plane than the node y.

- Exercies. We draw the Hasse digram for the poset $(2^{[3]}, \subset)$. Note that there should be 8 nodes as there are 8 subsets in $2^{[3]}$. Draw it again on your own.

- For any poset, we can express it by Hasse diagram. On the other hand, given a Hasse diagram, this diagram also defines a poset.

  The fact that $x \prec y$ if and only if there exist finite many elements $x_1, x_2, ..., x_k \in X$ such that $x \lhd x_1 \lhd x_2 \lhd \ldots \lhd x_k \lhd y$ now can be restated as: $x \prec y$ if and only if we can find a path in Hasse diagram from nod x to nod y, strictly from bottom to top.

- **Definition.** Let $P = (X, \preceq)$ be a poset.

  (1). For two distinct elements $x, y \in X$, if $x \prec y$ or $y \prec x$, then we say x, y are *comparable*; otherwise, x, y are *incomparable*.

  (2). Set $A \subset X$ is called an *antichain* (or *independent set*) of poset P, if any pair of distinct elements of A are incomparable. Denote $\alpha(P)$ to be the maximum over all antichains A of $|A|$, which is also called the *width* of P.

  (3). Set $A \subset X$ is called a *chain* of poset P, if any pair of distinct elements of A are comparable. Denote $\omega(P)$ to be the maximum over all chains A of $|A|$, which is also called the *height* of P.

  Understand above definitions in Hasse diagrams (see examples on page 55 of textbook). In particular, $\omega(P)$ means the maximal length of a path (directly from bottom to top) in its Hasse diagram.

- **Fact 1.** The set of minimal elements of poset $P = (X, \preceq)$ is always an antichain of P.

  Recall that an element $x \in X$ is minimal, if there is no other $y \in X$ such that $y \prec x$. This fact easily follows by definition and will play an important role in the proof of the following theorem.

- **Theorem.** For any poset $P = (X, \preceq)$, we have $\alpha(P) \cdot \omega(P) \geq |X|$.

- *Proof.* We inductively define a sequence of posets P_i, $1 \leq i \leq l$, as following. Let $P_1 = (X, \preceq_1)$ (here $\preceq_1 = \preceq$, so $P_1 = P$), and let M_1 be the set of all minimal elements of P_1. Now assume that $P_1, P_2, ..., P_k$ are all defined and let M_i be the set of all minimal elements of P_i. We use $\preceq_{k+1}$ to denote the ordering $\preceq$ resticted on set $X - \cup_{i=1}^{k} M_i$; let poset $P_{k+1} := (X - \cup_{i=1}^{k} M_i, \preceq_{k+1})$. We proceed the above inductive steps until $X = M_1 \cup M_2 \cup ... \cup M_l$.

  When the inductive step ends, we see that sets $M_1, ..., M_l$ form a partition of X. Moveover, each M_i is an antichain of poset P_i (by Fact 1) and thereby an antichain of poset P, therfore each $|M_i| \leq \alpha(P)$. Next we want to show that there exists a chain $x_1 \prec x_2 \prec ... \prec x_l$, where $x_i \in M_i$, which will be a consequence of the following claim.

  Claim: for any $x \in M_{k+1}$ for any $k = 1, ..., l-1$, there exists a $y \in M_k$ such that $y \prec x$. To see this, note that $x \in M_{k+1}$ implies: x is a minimal element of $P_{k+1} = (X - \cup_{i=1}^{k} M_i, \prec_{k+1})$ and $x \notin M_k$. Therefore x has a predecessor y in P_k but not in P_{k+1}, therefore $y \in X - \cup_{i=1}^{k-1} M_i$ and $y \notin X - \cup_{i=1}^{k} M_i$, implying that $y \in M_k$ (with $y \prec x$). This proves claim.

  Now we pick an element $x_l \in M_l$. Repeatly applying the claim, we see that there exists $x_i \in M_i$ such that $x_1 \prec x_2 \prec ... \prec x_{l-1} \prec x_l$. Therefore, $\omega(P) \geq l$.

  We get that $|X| = |\cup_{i=1}^{l} M_i| = \sum_{i=1}^{l} |M_i| \leq l \cdot \alpha(P) \leq \omega(P) \cdot \alpha(P)$. ∎

Eroős-Szekeres Theorem

- We proved that $\alpha(P) \cdot \omega(P) \geq |X|$ for any poset $P = (X, \preceq)$. We now see one nice application of this theorem.

- Definition. Consider a sequence $(x_1, x_2, ..., x_n)$ of real numbers of length n. A *subsequence* of length m is of the form $(x_{i_1}, x_{i_2}, ..., x_{i_m})$, where indices $i_1 < i_2 < ... < i_m$. This subsequence is *monotone*, if either $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_m}$ or $x_{i_1} \geq x_{i_2} \geq ... \geq x_{i_m}$.

  For example, in sequence 138795624, 18956 is a subsequence but not monotone.

- **Erdős-Szekeres' Theorem.** Any sequence $(x_1, x_2, ..., x_{n^2+1})$ of real numbers of length $n^2 + 1$ contains a montone subsequence of length $n + 1$.

- *Proof.* Given the sequence $(x_1, x_2, ..., x_{n^2+1})$, we define a poset $P = (X, \preceq)$, where $X = \{1, 2, ..., n^2 + 1\}$ and ordering $\preceq$ on set X is defined by:

  $$i \preceq j \text{ if and only if } i \leq j \text{ and } x_i \leq x_j.$$

  It is easy to verify that $P = (X, \preceq)$ indeed is a poset as $\preceq$ is reflexive, antisymmetric and transitive.

  Therefore, we have $\alpha(P) \cdot \omega(P) \geq |X| = n^2 + 1$, implying that either $\alpha(P) \geq n + 1$ or $\omega(P) \geq n + 1$.

  <u>Case 1</u>: we have $\omega(P) \geq n + 1$.

  Then we have a chain of P that $i_1 \preceq i_2 \preceq ... \preceq i_{n+1}$. By definition of $\preceq$, we get $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_{n+1}}$ and $i_1 \leq i_2 \leq ... \leq i_{n+1}$ (in fact $i_1 < i_2 < ... < i_{n+1}$ as they are disctinct). Therefore we find a monotone subsequence of length $n + 1$ (it is increasing).

  <u>Case 2</u>: we have $\alpha(P) \geq n + 1$.

We have an antichain $\{i_1, i_2, ..., i_{n+1}\}$. Without losing of generality, we assume that $i_1 < i_2 < ... < i_{n+1}$. Beacuse each pair $i_j < i_k$ are incomparable, we get $x_{i_j} > x_{i_k}$ whenever $j < k$. Therefore we get $x_{i_1} > x_{i_2} > ... > x_{i_{n+1}}$, which is a monotone subsequence of length $n + 1$ (it is strictly decreasing). This completes the proof of Erdős-Szekeres' Theorem. ∎

- Exercise. Prove the following generalization of Erdős-Szekeres' Theorem:

  For any integers $k, l \geq 2$, any sequence of real numbers of length $kl + 1$ contains either a strictly decreasing subsequence of length $l + 1$ or a increasing (not neccessnary strictly increasing) subsequence of length $k + 1$.

- Note that when $k = l = n$, the above statement is the same as Erdős-Szekeres' Theorem.

Ramsey's Theorem on Graphs

- We turn to study a new area: Ramsey theory, which is similar to Erdős-Szekeres' Theorem in the principle of "Order from disorder".

  Instead of sequences, now we focus on other combinatorical objects: graphs (complete graphs more explicitly).

- We will first look at a rather simple setting: A party of six.

  Fact 1. Suppose a party has six participants. Participants may know each other or not. Then, there must exist 3 participants who know each other or don't know each other.

  Proof. We contruct a complete graph on six vertices, each of which stands for one of the 6 participants. We then color the edges of this K_6 by colors blue and red in the following way: if i, j know each other, we color the edge (i, j) by blue; otherwise, color (i, j) by red. Therefore, what we want to show becomes that finding a triangle whose 3 edges are colored by the same color (blue or red).

  To see this, consider vertex 1 and its 5 incident edges. There must exist 3 incident edges of the same color. By the symmectry between blue and red, let us assume that edges $(1, 2), (1, 3), (1, 4)$ are blue. If one of edges $(2, 3), (2, 4), (3, 4)$ is colored by blue, say edge $(2, 3)$, then $1, 2, 3$ form a triangle with three blue edges. Otherwise, all three edges $(2, 3), (2, 4), (3, 4)$ are red, then $2, 3, 4$ form a triangle with edges all red. ∎

- Definition. A *r-edge-coloring* of K_n is a function $f : E(K_n) \to \{1, 2, ..., r\}$ which assigns one of the colors $1, 2, ..., r$ to each edge of K_n. If it is a 2-edge-coloring, usually we assume the colores are blue and red.

- Definition. Suppose the edges of a graph are colored by many colors. A k-clique (or K_k) is *monochromatic*, if all of its edges are colored by the same color. For example, if monochromatic K_k uses color blue, then we also call a *blue* K_k.

- Now it is easy to see that Fact 1 is equivalent to the following: any 2-edge-coloring of K_6 contains a monochromatic K_3 (a blue K_3 or red K_3).

  The coming fundamental theorem gives a generalization of the above fact.

- **Ramsey's Thorem.** (2-edge-coloring version)

  Let $k, l \geq 2$ be integers. There exists an integer N such that any 2-edge-coloring of K_N (with the colors blue and red) has a blue K_k or a red K_l.

- The main tool in the proof of Ramsey's theorem is Pigeonhole Principle.

  Pigeonhole Principle. Let $A_1, A_2, ..., A_k$ be disjoint sets which form a partition of the groud set X, where $|X| = 1 + \sum_{i=1}^{k}(a_i - 1)$. Then there exists some i such that $|A_i| \geq a_i$.

- **The proof of Ramsey's theorem.** We will show that N can be picked as $N = \binom{k+l-2}{k-1}$, where k, l are symmetric in the expression of N, as $\binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}$.

  We proceed by induction on the sum of $k+l$ to prove the statement: any 2-edge-coloring of $K_{\binom{k+l-2}{k-1}}$ has a blue K_k or a red K_l. The base case ($k = l = 2$) is trivial, as $N = \binom{k+l-2}{k-1} = \binom{2}{1} = 2$.

  Now we consider $k, l \geq 2$ and assume that the statement holds for any pairs (k', l') whose sum is smaller than $k+l$ (in particular, the statement holds for pair $(k-1, l)$ as well as for pair $(k, l-1)$). For the purpose of presentation, write $N := \binom{k+l-2}{k-1}$, $N_1 := \binom{(k-1)+l-2}{(k-1)-1} = \binom{k+l-3}{k-2}$ and $N_2 := \binom{k+(l-1)-2}{k-1} = \binom{k+l-3}{k-1}$. Therefore, we get $N_1 + N_2 = N$, beacuse of the identity $\binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a}{b}$.

  Consider an arbitrary 2-edge-coloring of K_N. Similar to the proof of "a party of six", we pick a vertex $u \in V$ (note that this u can be arbitrary!). Then we partition $V - \{u\}$ into two disjoint sets A, B, where

  $$A = \{x \in V - \{u\} : \text{edge } xu \text{ is colored by blue}\}$$

  and

  $$B = \{x \in V - \{u\} : \text{edge } xu \text{ is colored by red}\}.$$

  Therefore, $|A|+|B|+1 = |V| = N = N_1+N_2$, implying that $|A|+|B| = 1+(N_1-1)+(N_2-1)$. By Pigeonhold principle, we have either $|A| \geq N_1$ or $|B| \geq N_2$.

  <u>Case 1</u>: $|A| \geq N_1 = \binom{(k-1)+l-2}{(k-1)-1}$.

  The vertices of A contains a complete graph $K_{\binom{(k-1)+l-2}{(k-1)-1}}$, whose edges are colored by blue and red. By induction on this graph for the pair $(k-1, l)$, set A has either a blue K_{k-1} or a red K_l. If A has a blue K_{k-1}, notice that all edges between u and A are blue, then this blue K_{k-1} plus vertex u give us a blue K_k. Therefore, in Case 1, there must exist a blue K_k or a red K_l, as wanted.

  <u>Case 2</u>: $|B| \geq N_2 = \binom{k+(l-1)-2}{k-1}$.

  Similar to Case 1, the vertices of B contains a complete graph $K_{\binom{k+(l-1)-2}{k-1}}$. By induction on this graph for the pair $(k, l-1)$, set B has either a blue K_k or red K_{l-1}. Note that all edges between u and B are red, if B has a red K_{l-1}, then this K_{l-1} plus vertex u give us a red K_l. Therefore, again in Case 2, there must exist a blue K_k or a red K_l. This finishes the proof of Ramsey's theorem. ∎

<u>Graph Ramsey Numbers</u>

- We proved the Ramsey's Theorem that for any integer $k, l \geq 2$, there exists an integer N such that any 2-edge-coloring of K_N has a blue K_k or a red K_l. In fact, we show that N can be $N = \binom{k+l-2}{k-1}$.

- **Definition.** For any integers $k, l \geq 2$, the *Ramsey number* $R(k, l)$ denotes the **smallest** integer N such that any 2-edge-coloring of K_N has a blue K_k or red K_l.

- Let us try to understand the following inequalities:

  (i) $R(k, l) \leq L$ means that any 2-edge-coloring of K_L has a blue K_k or red K_l;

  (ii) $R(k, l) > M$ means that there exists a 2-edge-coloring of K_M containing neither blue K_k nor red K_l.

- It is very hard to find the exact value of $R(k, l)$, even for small k, l (for example it is not known what is $R(5, 5)$). Instead, we will estimate $R(k, l)$ by providing lower/upper bounds. Recall the meanings of $R(k, l) \leq L$ and $R(k, l) > M$.

- **Fact 1.** We have $R(k, l) \leq \binom{k+l-2}{k-1}$.

- **Fact 2.** $R(k, l) = R(l, k)$.

- **Fact 3.** $R(2, l) = l$ and $R(k, 2) = k$ for all $k, l \geq 2$.

- **Fact 4.** $R(3, 3) = 6$.

  $R(3, 3) \leq 6$ follows by the "party of six" problem; $R(3, 3) > 5$ follows by a 2-edge-coloring of K_5 we constructed in class.

- **Fact 5.** $R(3, 4) = 9$.

  We show that $R(3, 4) > 8$ by constructing a graph on 8 vertices which contains no triangle and no independent set of size 4. Note that an independent set is a graph which has no edge at all.

  And the proof of $R(3, 4) \leq 9$ in fact can be generalized to the following:

  Theorem. If Ramsey numbers $R(k-1, l)$ and $R(k, l-1)$ are both even, then we have $R(k, l) \leq R(k-1, l) + R(k, l-1) - 1$.

  Note that this is stronger than one of the homework problems.

- **Fact 6.** It is also known that $R(4, 4) = 18, R(4, 5) = 25$ and $43 \leq R(5, 5) \leq 49$. We mention them without proofs.

- Definition. When $k = l$, the Ramsey number $R(k, k)$ is called the *diagonal Ramsey number*.

- In next week, we will show a lower bound of $R(k, k)$.

  Note that to show $R(k, k) > n$, we need to construct a 2-edge-coloring of K_n such that it has no monochromatic K_k. We will achieve this by probabilistic tools (without explicitly giving the construction of the desired 2-edge-coloring).